Functions

Part Two

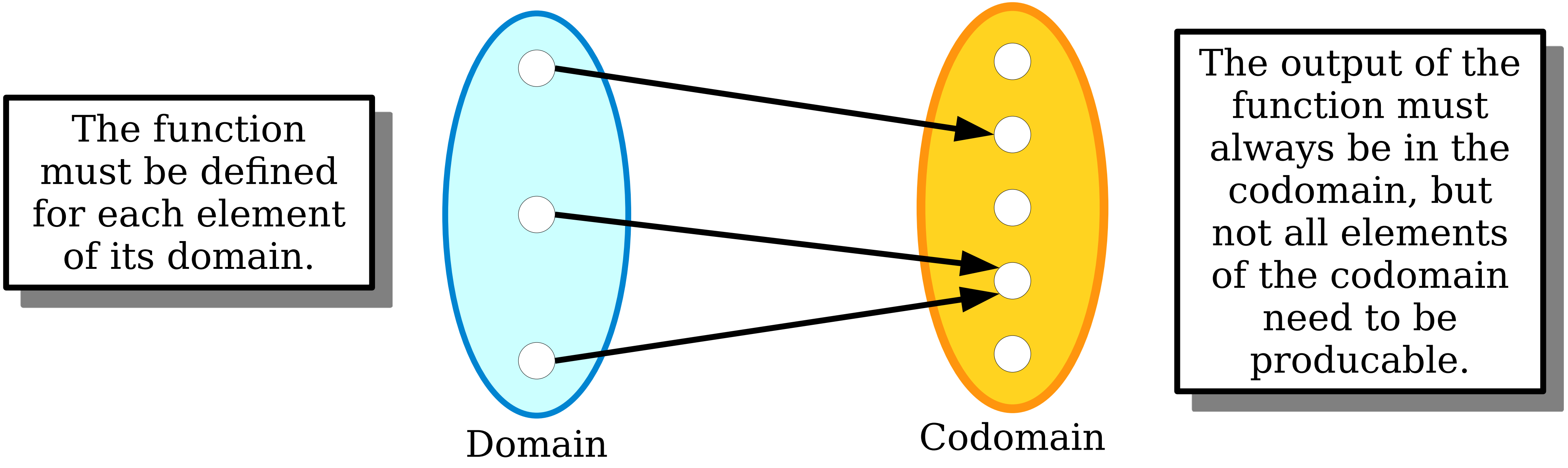
Outline for Today

- ***Recap from Last Time***
  - Where are we, again?
- ***Injections and Surjections***
  - Two useful classes of functions.
- ***A Proof About Birds***
  - Trust me, it's relevant.
- ***Assuming vs Proving***
  - Two different roles to watch for.
- ***Connecting Function Types***
  - Relating the topics from last time.

Recap from Last Time

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.
- We write $\boldsymbol{f : A \to B}$ to indicate that *f* is a function whose domain is *A* and whose codomain is *B*.



Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- For example, $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = -x$ is an involution.

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if different inputs always map to different outputs.
  - A function with this property is called an ***injection***.
- Formally, $f : A \to B$ is an injection if this FOL statement is true:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different"*)

- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same"*)

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every output, there's an input that produces it."*)

- A function with this property is called a ***surjection***.

	To ***prove*** that this is true…	
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	
$A \to B$	Assume A is true, then prove B is true.	
$A \land B$	Prove A. Then prove B.	
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	
$\neg A$	Simplify the negation, then consult this table on the result.	

New Stuff!

A Proof About Birds

Theorem: If all birds can fly,
then all herons can fly.

Theorem: If all birds can fly, then all herons can fly.

Given the predicates

$Bird(b)$, which says b is a bird;
$Heron(h)$, which says h is a heron; and
$CanFly(x)$, which says x can fly,

translate the theorem into first-order logic.

Answer at
https://pollev.com/cs103aut23

Theorem: If all birds can fly, then all herons can fly.

Given the predicates

Bird(b), which says b is a bird;
Heron(h), which says h is a heron; and
CanFly(x), which says x can fly,

translate the theorem into first-order logic.

$$(\forall b.\ (Bird(b) \to CanFly(b))) \to (\forall h.\ (Heron(h) \to CanFly(h)))$$

All birds can fly

All herons can fly

	To ***prove*** that this is true...	
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	
$A \to B$	Assume A is true, then prove B is true.	
$A \land B$	Prove A. Then prove B.	
	Either prove $\neg A \to B$ or	

$$(\forall b.\ (Bird(b) \to CanFly(b))) \to (\forall h.\ (Heron(h) \to CanFly(h)))$$

All birds can fly

All herons can fly

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Answer at

https://pollev.com/cs103aut23

Which makes more sense as the next step in this proof?

1. Consider an arbitrary bird b.
2. Consider an arbitrary heron h.

$$(\forall b.\ (Bird(b) \to CanFly(b))) \to (\forall h.\ (Heron(h) \to CanFly(h)))$$

All birds can fly

All herons can fly

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Which makes more sense as the next step in this proof?

1. Consider an arbitrary bird b.
2. Consider an arbitrary heron h.

$$(\forall b.\ (Bird(b) \to CanFly(b))) \to (\forall h.\ (Heron(h) \to CanFly(h)))$$

All birds can fly

All herons can fly

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Consider an arbitrary bird b. Since b is a bird, b can fly. *[and now we're stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{All birds can fly}} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{All herons can fly}}$$

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Which makes more sense as the next step in this proof?

1. Consider an arbitrary bird b.
2. Consider an arbitrary heron h.

$$(\forall b.\ (Bird(b) \to CanFly(b))) \to (\forall h.\ (Heron(h) \to CanFly(h)))$$

All birds can fly

All herons can fly

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Consider an arbitrary heron h. We will show that h can fly. To do so, note that since h is a heron we know h is a bird. Therefore, by our earlier assumption, h can fly. ■

$$(\forall b.\ (Bird(b) \to CanFly(b))) \to (\forall h.\ (Heron(h) \to CanFly(h)))$$

All birds can fly

All herons can fly

Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we ***assumed*** all birds can fly.
  - Here, we ***proved*** all herons can fly.
- Statements behave differently based on whether you're assuming or proving them.

$$(\forall b.\ (Bird(b) \to CanFly(b))) \to (\forall h.\ (Heron(h) \to CanFly(h)))$$

We never introduce a variable *b*.

We introduce a variable *h* almost immediately.

Proving vs. Assuming

- To ***prove*** the universally-quantified statement

$$\forall x.\ P(x)$$

  we introduce a new variable x representing some arbitrarily-chosen value.
- Then, we prove that $P(x)$ is true for that variable x.
- That's why we introduced a variable h in this proof representing a heron.

$$(\forall b.\ (Bird(b) \to CanFly(b))) \to (\forall h.\ (Heron(h) \to CanFly(h)))$$

We never introduce a variable b.

We introduce a variable h almost immediately.

Proving vs. Assuming

- If we ***assume*** the statement

$$\forall x.\ P(x)$$

  we ***do not*** introduce a variable x.
- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h, our heron, can fly.

$$(\forall b.\ (Bird(b) \to CanFly(b))) \to (\forall h.\ (Heron(h) \to CanFly(h)))$$

We never introduce a variable b.

We introduce a variable h almost immediately.

	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Connecting Function Types

Types of Functions

- We now have three special types of functions:
  - ***involutions***, functions that undo themselves;
  - ***injections***, functions where different inputs go to different outputs; and
  - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \quad \to \quad (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

f is an involution. → *f* is surjective.

Theorem: For any function $f : A \to A$,
if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \ \to \ (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

Assume this. Prove this.

$$(\forall b.\ (Bird(b) \to CanFly(b))) \ \to \ (\forall h.\ (Heron(h) \to CanFly(h)))$$

Assume this. Prove this.

Theorem: For any function $f : A \to A$,
if f is an involution, then f is surjective.

$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$

Assume this.

Prove this.

If you *assume* this is true...

Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

Assume this.

Prove this.

Since we're assuming this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$

Ass

There's a universal quantifier up front. Since we're *proving* this, we'll pick an arbitrary $b \in A$.

Prove this.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$

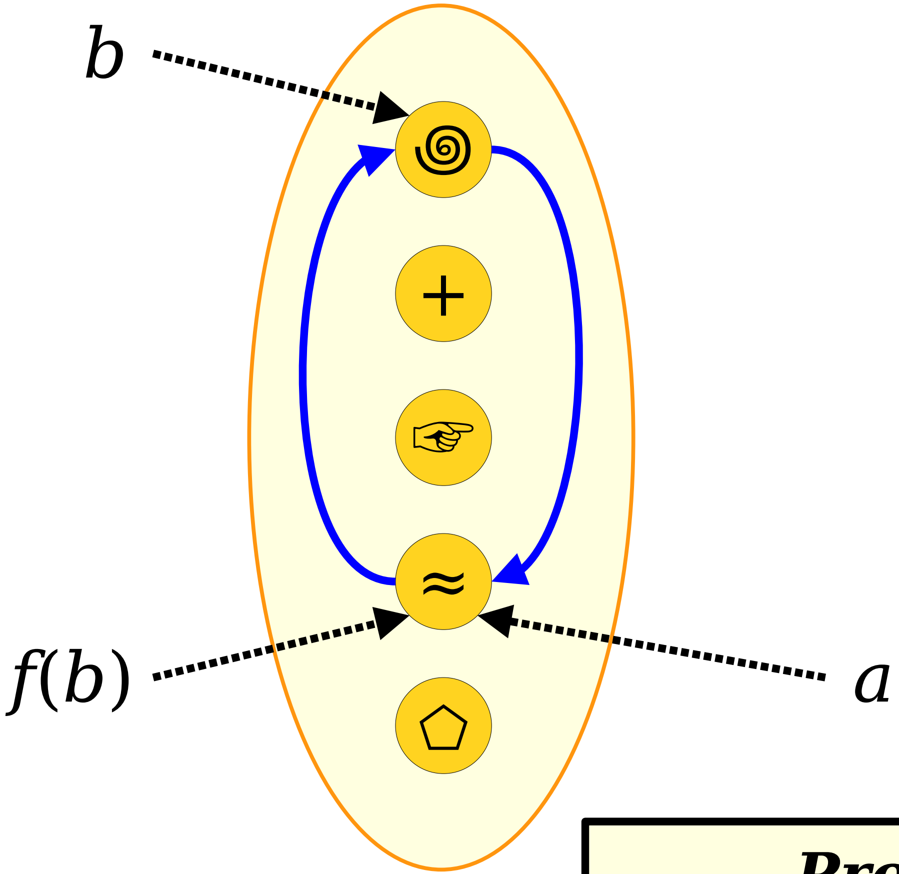
Now, we hit an existential quantifier. Since we're *proving* this, we need to find a choice of $a \in A$ where this is true.

Prove this.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.



Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Proof: Pick any involution $f : A \to A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$. We need to show that there is an $a \in A$ where $f(a) = b$.

Specifically, pick $a = f(b)$. This means that $f(a) = f(f(b))$, and since f is an involution we know that $f(f(b)) = b$. Putting this together, we see that $f(a) = b$, which is what we needed to show. ■

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Proof: Pick any involution $f : A \to A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$. We need to show that there is an $a \in A$ where $f(a) = b$.

Specifically, pick $a = f(b)$. This means that $f(a) = f(f(b))$, and since f is an involution we know that $f(f(b)) = b$. Putting this together, we see that $f(a) = b$, which is what we needed to show. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

The Two-Column Proof Organizer

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

Theorem: Let $f : A \to A$ be an involution.
Then f is injective.

What We're Assuming	***What We Need to Prove***
$f : A \to A$ is an involution.	f is injective.
$\forall z \in A.\ f(f(z)) = z.$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$

We're *assuming* this universally-quantified statement, so we won't introduce a variable for what's here.

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

What We're Assuming	***What We Need to Prove***
$f : A \to A$ is an involution.	f is injective.
$\forall z \in A.\ f(f(z)) = z.$	~~$\forall a_1 \in A.\ \forall a_2 \in A.$~~ $(f(a_1) = f(a_2) \to a_1 = a_2)$
$a_1 \in A$	
$a_2 \in A$	

We need to prove this implication. So we assume the antecedent and prove the consequent.

Theorem: Let $f : A \to A$ be an involution.
Then f is injective.

What We're Assuming

$f : A \to A$ is an involution.

$\forall z \in A.\ f(f(z)) = z.$

$a_1 \in A$

$a_2 \in A$

$f(a_1) = f(a_2)$

$f(f(a_1)) = f(f(a_2))$

$f(f(a_1)) = a_1$

$f(f(a_2)) = a_2$

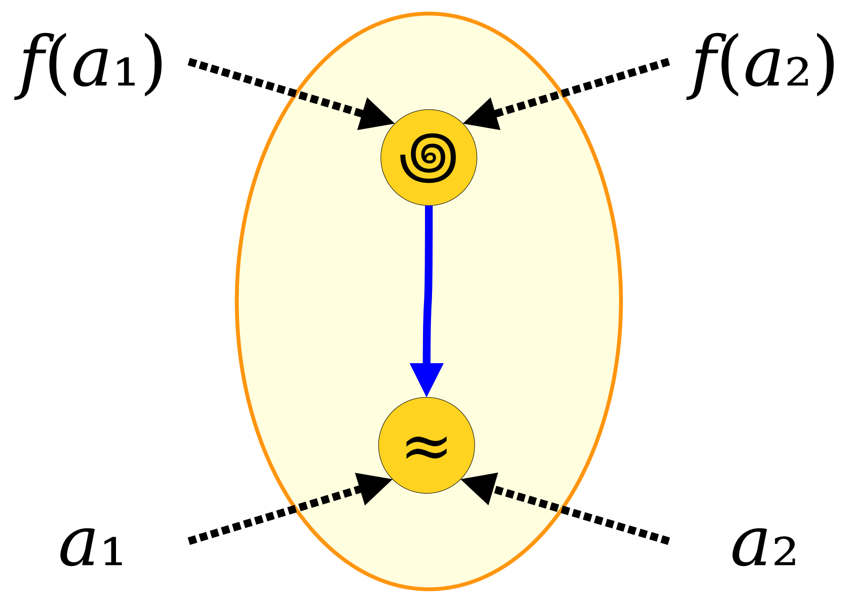
What We Need to Prove

f is injective.

$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to$

$a_1 = a_2$

$)$



Theorem: Let $f : A \to A$ be an involution. Then f is injective.

Proof: Choose any $a_1, a_2 \in A$ where $f(a_1) = f(a_2)$. We need to show that $a_1 = a_2$.

Since $f(a_1) = f(a_2)$, we know that $f(f(a_1)) = f(f(a_2))$. Because f is an involution, we see $a_1 = f(f(a_1))$ and that $f(f(a_2)) = a_2$. Putting this together, we see that

$$a_1 = f(f(a_1)) = f(f(a_2)) = a_2,$$

so $a_1 = a_2$, as needed. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

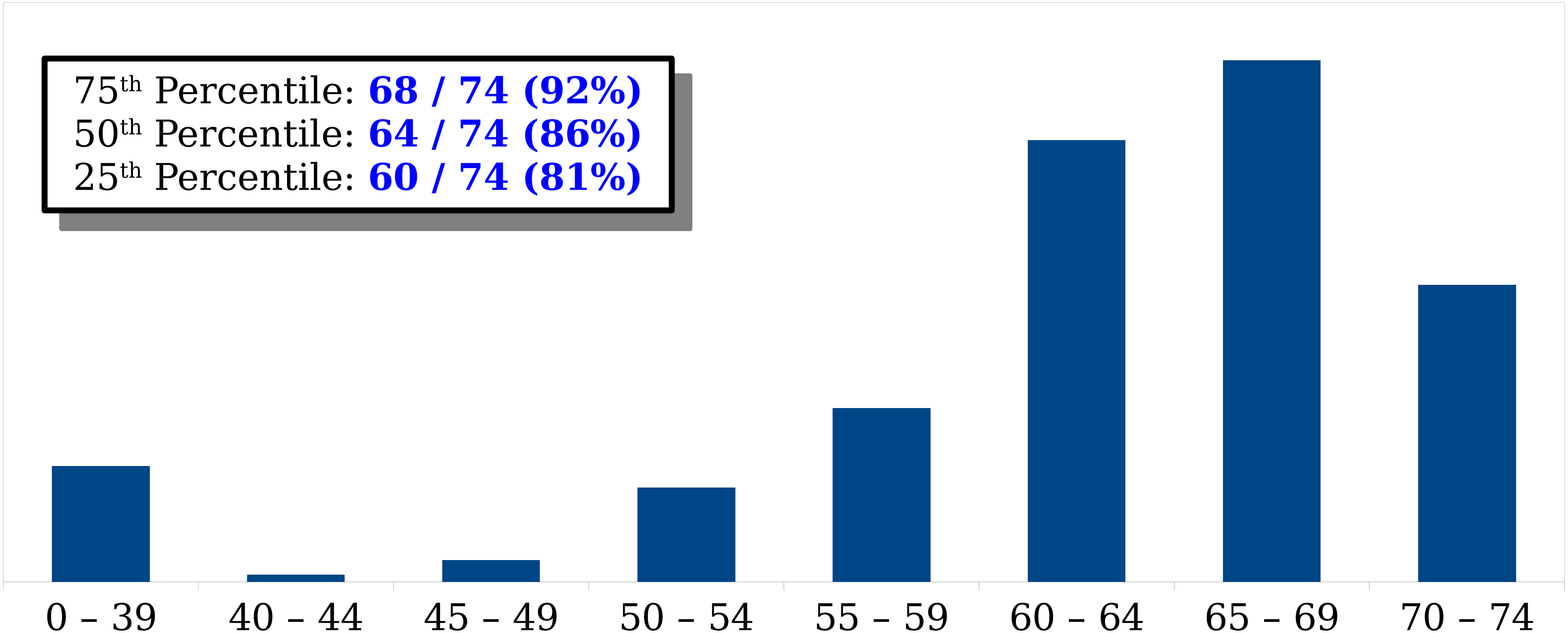
Time-Out for Announcements!

Problem Set One Graded

- Your wonderful TAs have finished grading Problem Set One.
- Grades and feedback are up on the Gradescope.
- Solutions are available online on the course website (visit the page for PS1 to get the link).

Problem Set One Graded

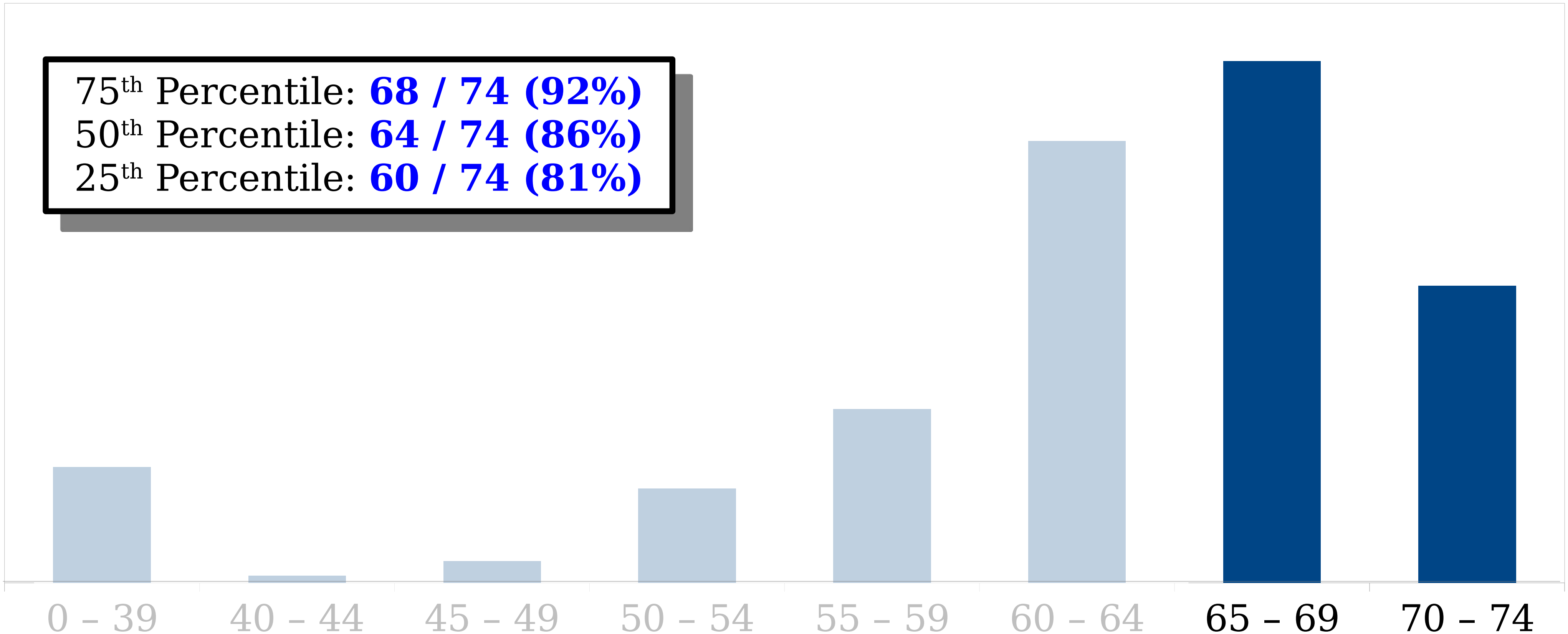
75^{th} Percentile: **68 / 74 (92%)**
50^{th} Percentile: **64 / 74 (86%)**
25^{th} Percentile: **60 / 74 (81%)**



Pro tips when reading a grading distribution:

1. Standard deviations are ***unhelpful and discouraging***. Ignore them.
2. The average score is a ***unhelpful***. Ignore it.
3. Raw scores are ***unhelpful and discouraging***. Ignore them.

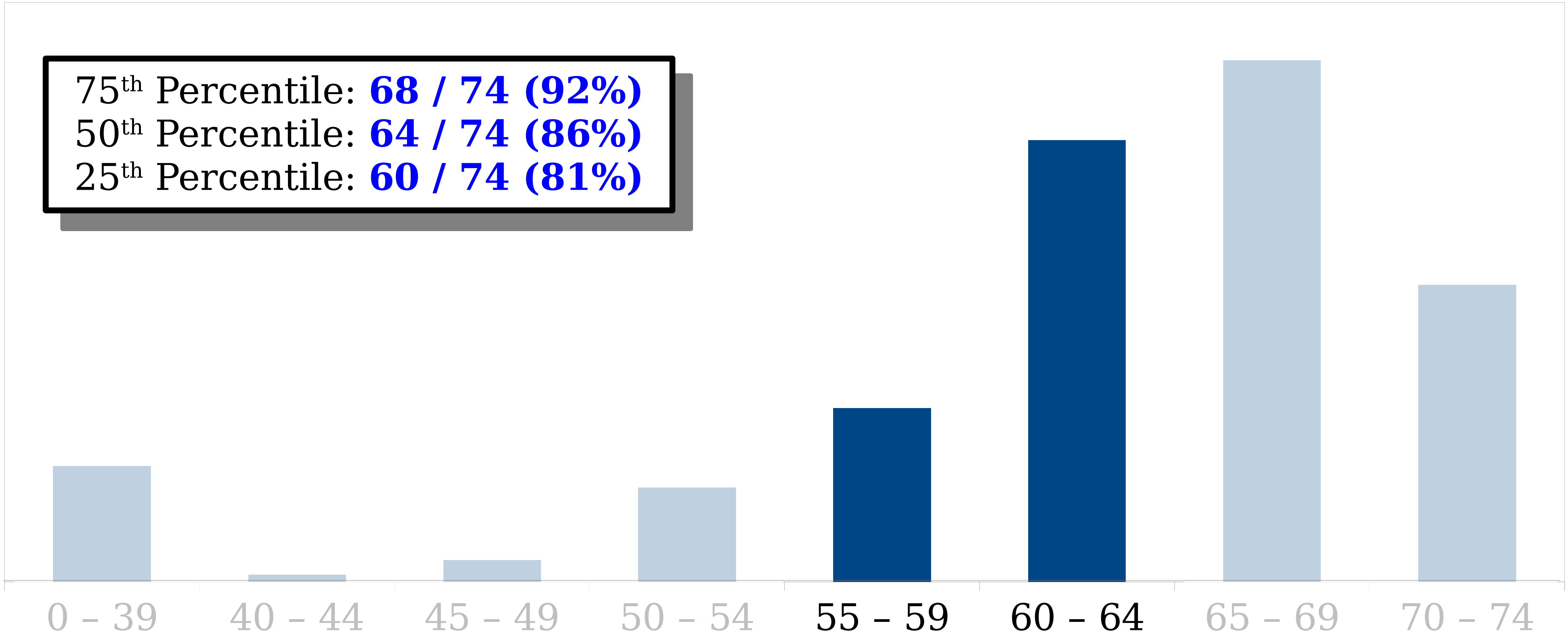
Problem Set One Graded



75th Percentile: **68 / 74 (92%)**
50th Percentile: **64 / 74 (86%)**
25th Percentile: **60 / 74 (81%)**

"Great job! Look over your feedback for some tips on how to tweak things for next time."

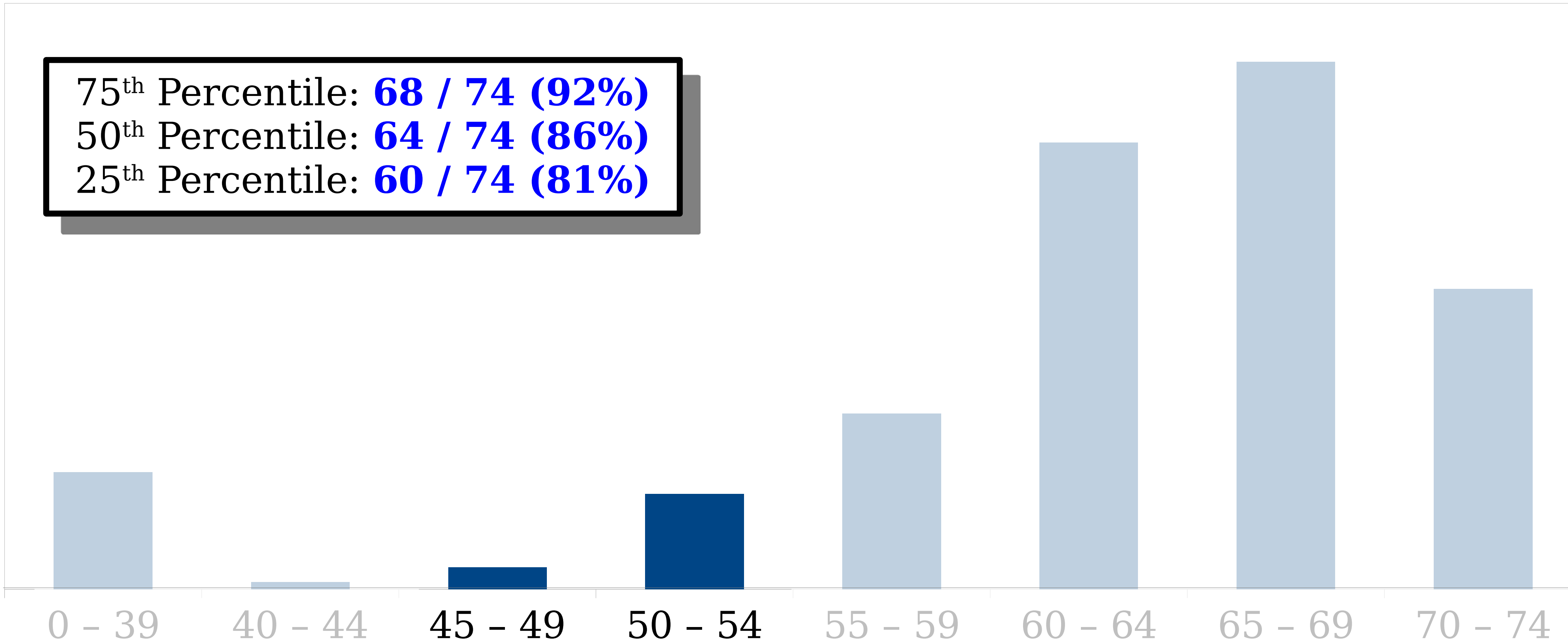
Problem Set One Graded



"You're almost there! Review the feedback on your submission and see what to focus on for next time."

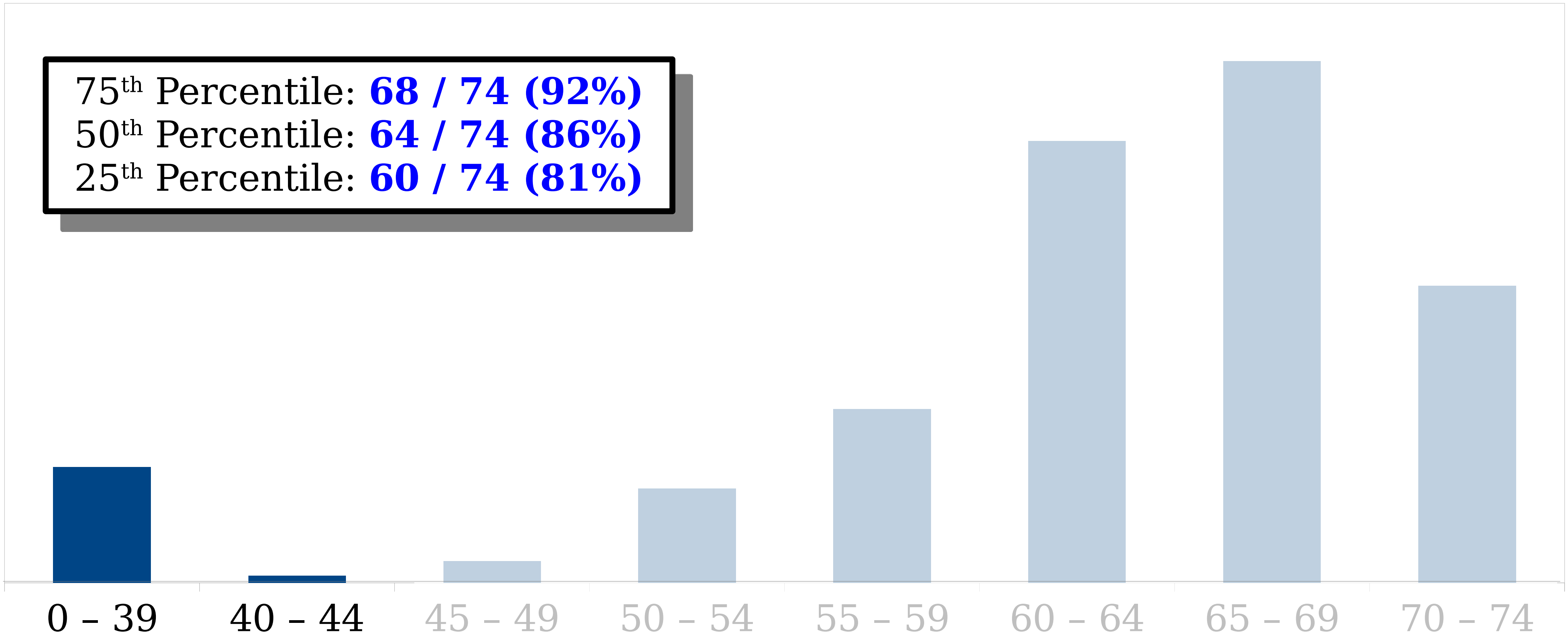
Problem Set One Graded

75th Percentile: **68 / 74 (92%)**
50th Percentile: **64 / 74 (86%)**
25th Percentile: **60 / 74 (81%)**



"You're on the right track, but there are some areas where you need to improve. Review your feedback and ask us questions when you have them."

Problem Set One Graded



"Looks like something hasn't quite clicked yet. Get in touch with us and stop by office hours to get some extra feedback and advice. Don't get discouraged – you can do this!"

What Not to Think

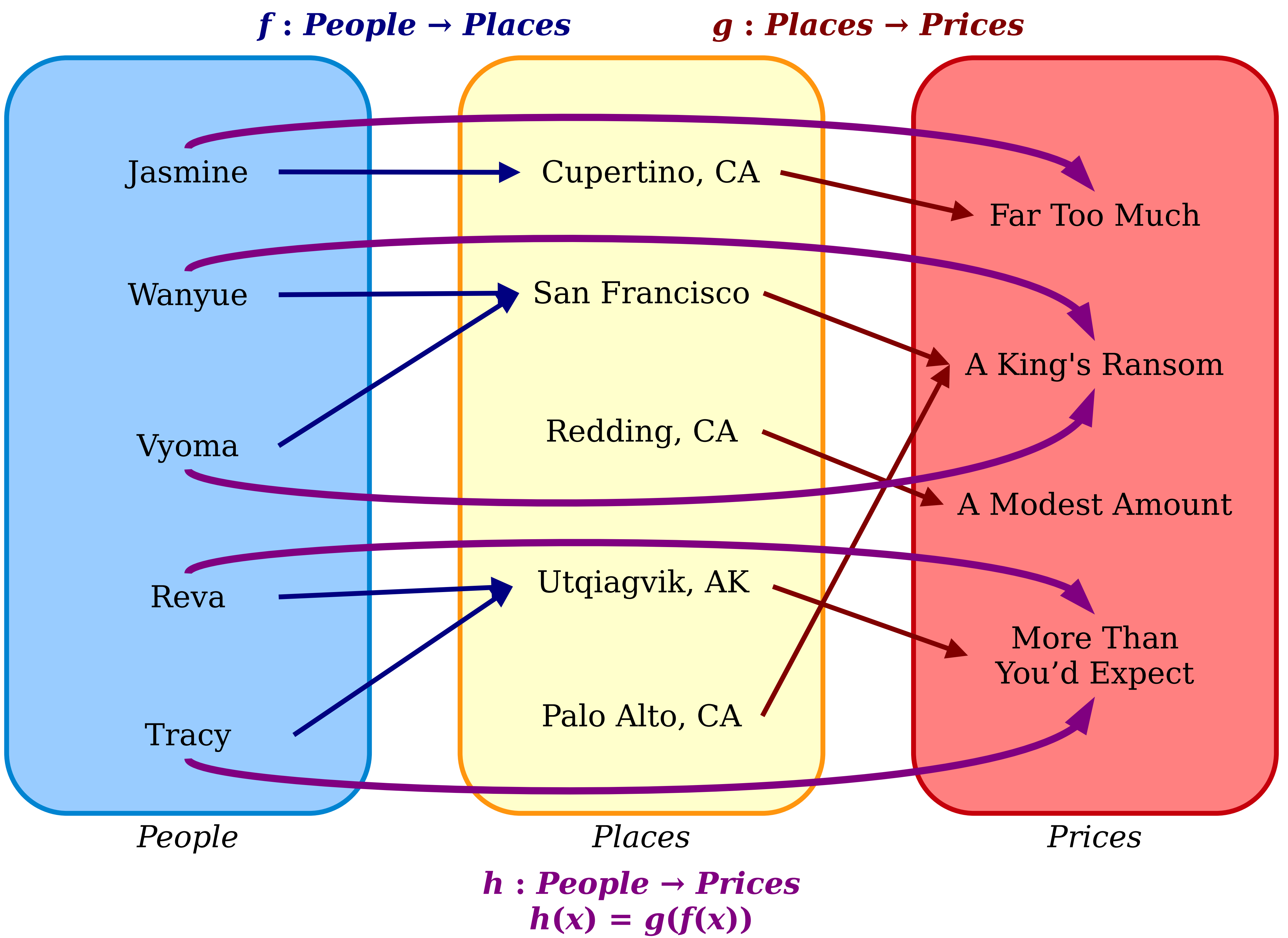
- “Well, I guess I’m just not good at math.”
  - For most of you, this is your first time doing any rigorous proof-based math.
  - Don’t judge your future performance based on a single data point.
  - Life advice: think about download times.
  - Life advice: have a growth mindset!
- “Hey, I did above the median. That’s good enough.”
  - There’s always some area where you can improve. Take the time to see what that is.

Regrade Requests

- We’re human. We make mistakes. And we’re happy to correct them!
- Regrades will open on Gradescope 48 hours after grades are released. They close one week after grades are released.
- Notes on regrades:
  - Please be civil. We make mistakes. We’re happy to correct them.
  - We have to grade what you submitted; we can’t take any clarifications into account during regrades.
  - Regrades are for where we made deductions we shouldn’t have, rather than for the magnitude of deductions.

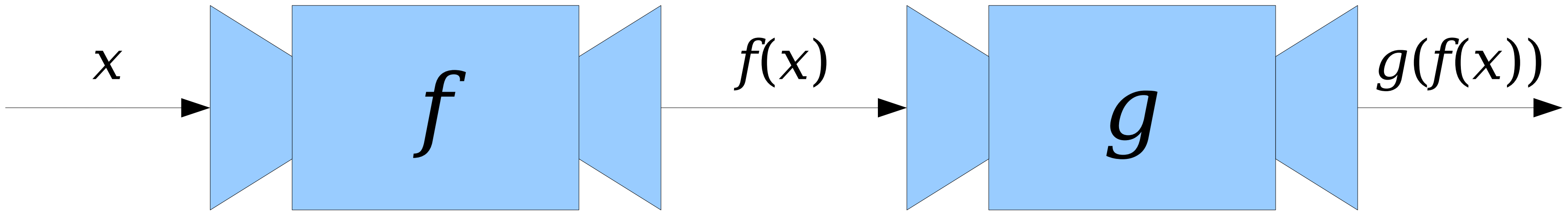
Back to CS103!

Function Composition


f : People → Places
g : Places → Prices
Jasmine
Wanyue
Vyoma
Reva
Tracy
Cupertino, CA
San Francisco
Redding, CA
Utqiagvik, AK
Palo Alto, CA
Far Too Much
A King's Ransom
A Modest Amount
More Than You’d Expect
People
Places
Prices
h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

> The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

Properties of Composition

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$$

We're *assuming* these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	*What We Need to Prove*
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection.	
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

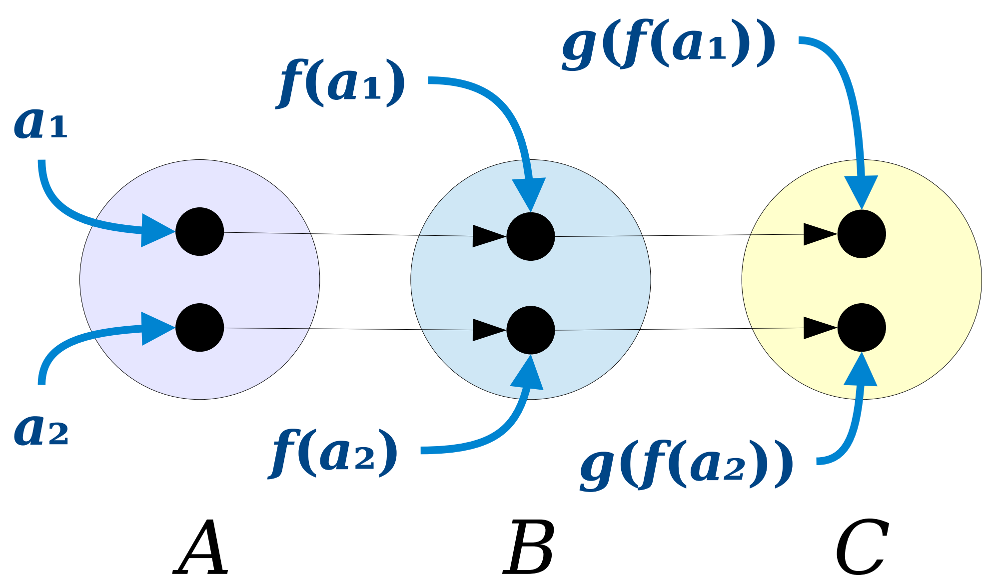
Now we're looking at an implication. Let's *assume* the antecedent and *prove* the consequent.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection.	$g(f(a_1)) \neq g(f(a_2))$
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection.	$g(f(a_1)) \neq g(f(a_2))$
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

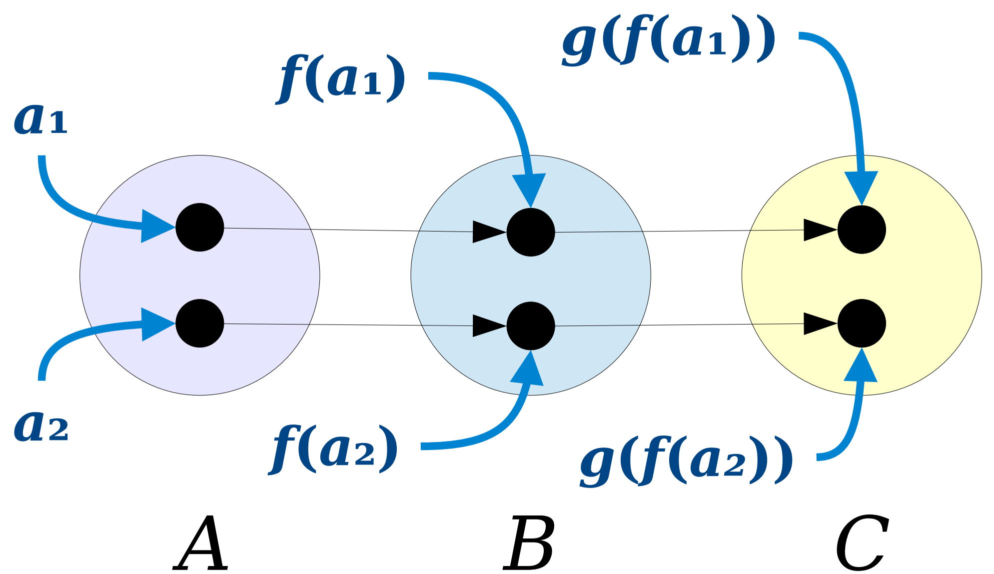


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

Great exercise: Repeat this proof using the other definition of injectivity.

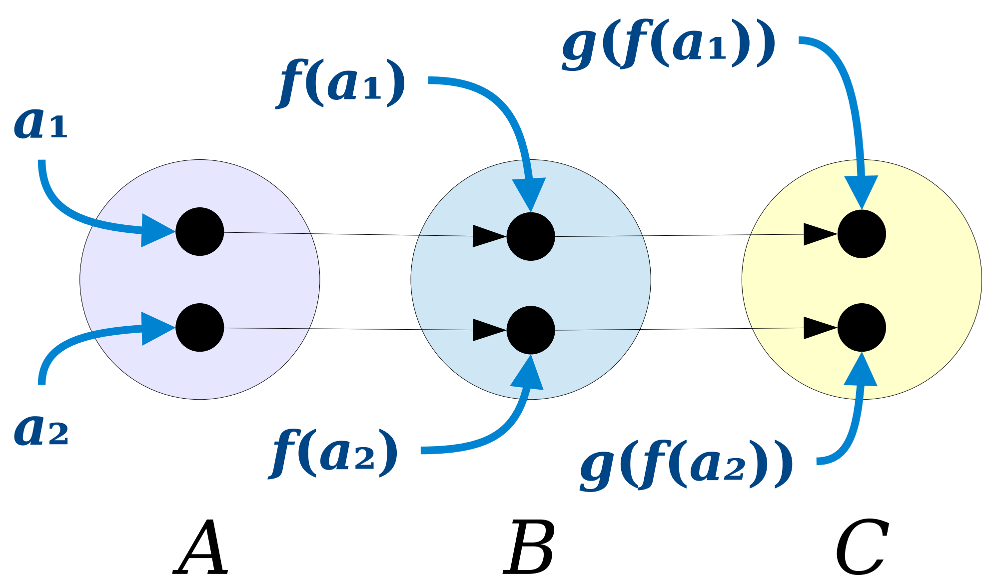


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.



Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

Proof: In the appendix!

Major Ideas From Today

- Proofs involving first-order definitions are heavily based on the structure of those definitions, yet FOL notation itself does *not* appear in the proof.
- Statements behave differently based on whether you're ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.

	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next Time

- ***Set Theory Revisited***
  - Formalizing our definitions.
- ***Images and Preimages***
  - Applying functions to sets of items.

Appendix: Additional Function Proofs

Proof: Composing surjections yields a surjection.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$. Then we see that

$$g(f(a)) = g(b) = c,$$

which is what we needed to show. ■